Chapter 4. The Binomial Theorem and Related Identities

1. The Binomial Theorem.

Theorem. For all nonnegative integers n

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: Consider the product of n sums

$$(x+y)^n = (x+y)(x+y)\ldots(x+y)$$

We can compute this product by taking one summand (x or y) from each parentheses, multiply them together, then repeat this in all 2^n possible ways and add up the results.

We get a product equal $x^k y^{n-k}$ each time we take k summands (out of n) equal to x. There are exactly $\binom{n}{k}$ such products. $\square$

We usually call the expression $\binom{n}{k}$ as **binomial coefficient.**

Example 4.1. Prove that the alternating sum of binomial coefficients is equal to 0

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint. Apply the Binomial Theorem for $x=-1, y=1$.

Example 4.2. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Sol 1. Apply the Binomial Theorem for $x=y=1$.

Sol 2: The RHS is the number of subsets of $[n]$.
Classify the subsets of $[n]$ into n classes based on the sizes. The number of k-subsets is $\binom{n}{k}$. Thus LHS counts the sum of numbers of elements in all n classes of subsets, which is exactly the total number of subsets, i.e. 2^n. □

Example 4.3. Prove that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof: RHS counts $(k+1)$-subsets of $[n+1]$.
We will partition this set of $(k+1)$-subsets into two part.

① $(k+1)$-subsets containing $(n+1)$. By removing elemt $n+1$ from each of this subsets, we get k-subsets of $[n]$. Thus there are $\binom{n}{k}$ such $(n+1)$-subsets

(2) $(k+1)$-subsets not containing $n+1$. Thus, they are actually $(k+1)$-subsets of $[n]$. There are $\binom{n}{k+1}$ $(k+1)$-subsets of this type.

Summing up $\binom{n}{k}$ and $\binom{n}{k+1}$ we get the total $(k+1)$-subsets of $[n+1]$, which is equal the RHS $\square$

The triangular array below is called **Pascal triangle**

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

The array canbe defined as follows:

+ Each row starts and ends by 1.
+ An entry on row $n+1$, that are not the first and the last entry, is equal to the sum of two closest entries on the row n.

It is easy to prove (by induction and Example 4.3) that the kth entry on row n is exactly $\binom{n}{k}$.

Example 4.4

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

Proof: Similar to **Example 4.3.**
or we can apply Example 4.3 repeatedly. $\square$

This example means that each entry in the Pascal triangle is equal to the sum of all entries above it on the SW-NE diagonal next to it on the left.

Example 4.5.

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Note that it is not even obvious why

$$\sum_{k=1}^{n} k\binom{n}{k} \Big/ 2^{n-1} \quad \text{is an integer.}$$

Proof. Both sides count the ways to choose a committee among n people, then to choose a president from the committe.

LHS: we choose a k-member committee in $\binom{n}{k}$ ways, then choose it president in k ways.

RHS: we choose president first, in n ways, then choose other members of the committee from $n-1$ remaining in 2^{n-1} ways. $\square$

Alternative proof:

Let $y=1$ in the Binomial Theorem:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take derivative both sides, we have

$$n(x+1)^{n-1} = \sum_{k=1}^{n} k \cdot \binom{n}{k} x^{k-1}$$

Then the indentity follows by letting $x=1$. $\square$

Example 4.6. (Vandermonde's identity)

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}.$$

Proof:

LHS counts the k-subets of $[n+m]$.

RHS counts the same. We can first choose i elements from $[n]$ in $\binom{n}{i}$ ways, then choose the remaing $k-i$ elements from set $\{n+1, \dots, n+m\}$ in $\binom{m}{k-i}$ ways. $\square$

Consider elements in a row of the Pascal triangle. The entries are increasing from the leftmost to the center, and decreasing from the center to the right most.

Theorem 4.7. For all $k \leq \frac{n-1}{2}$, we have

$$\binom{n}{k} \leq \binom{n}{k+1}.$$

The identity holds if and only if $n = 2k+1$.

Proof.

NTS

$$\frac{n!}{k!\,(n-k)!} \leq \frac{n!}{(k+1)!\,(n-k+1)!}$$

$$\iff \quad \frac{1}{n+k} \leq \frac{1}{k+1}$$

$$\iff \quad n+k \leq k+1$$

$$\iff \quad 2k \leq n-1$$

$$\iff \quad k \leq \frac{n-1}{2} \qquad \square$$

Defn: **Lattice path** in the plane is a sequence

$$P = ((x_0, y_0), (x_1, y_1), \dots, (x_k, y_k))$$

where x_i's and y_i's are integers, and for $i \geq 1$, either $(x_i, y_i) = (x_{i-1}+1, y_{i-1})$ or $(x_{i-1}, y_{i-1}+1)$. We say P is a path from (x_0, y_0) to (x_k, y_k).

Example 4.8. How many lattice path from $(0,0) \to (a, b)$?

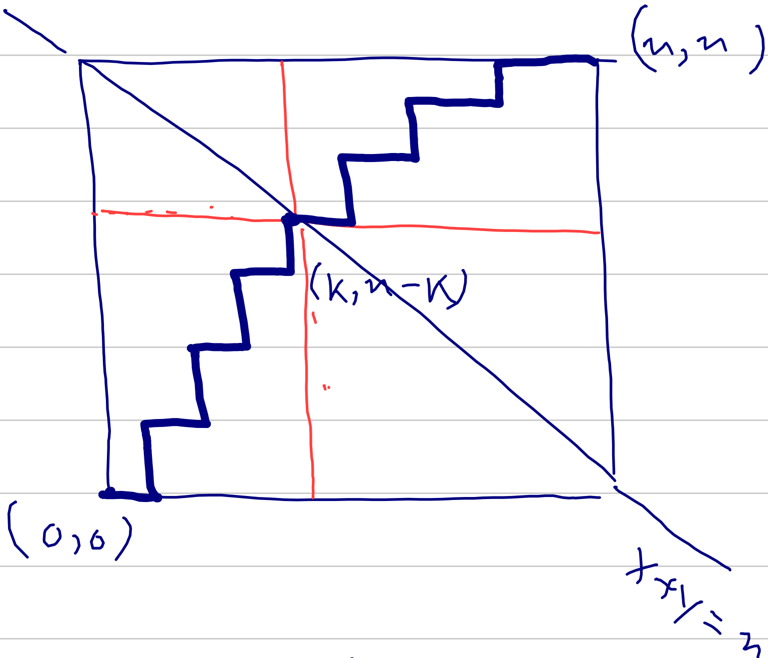
This is usually mentioned as the "Manhattan problem".

Answer $\binom{a+b}{a} = \binom{a+b}{b}$.

Example 4.9. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof:



RHS counts lattice path from $(0,0)$ to (n,n).

Each such path can be partitioned into two parts. The portion from $(0,0)$ to its intersection with the line $x+y=n$, and the remaining. Thus we can construct any path from $(0,0) \to (n,n)$ by gluing two paths: a path from $(0,0)$ to $(k, n-k)$ (for $k = 0, 1, .., n$), and a path from $(k, n-k)$ to (n,n).

There are both $\binom{n}{k}$ ways to choose those paths $\Rightarrow$ LHS also counts paths from $(0,0)$ to (n,n).

Example 4.10. Prove Example 4.4 and Example 4.6 by lattice paths.

2. The Multinomial Theorem.

Defn: Let $n = \sum_{i=1}^{n} a_i$, where n and $a_1, a_2, \dots, a_k$ are nonnegative integer's. We define

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1!\, a_2! \cdots a_k!},$$

called "multinomial coefficients".

Theorem (Multinomial Theorem)

For all nonnegative integers n and k, we have

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{a_1 + a_2 + \cdots + a_k = n} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}$$

Proof. Multiply out the LHS as

$$(x_1 + \cdots + x_k)(x_1 + \cdots + x_k) \cdots (x_1 + \cdots + x_k)$$

there are total k^n terms of form

$$y_1 y_2 \cdots y_n$$

where $y_i \in \{x_1, \dots, x_k\}$. Combine identical factors y_i's, we can write each term as

$$x_1^{a_1} x_2^{a_2} \cdots x_K^{a_K} \qquad (a_1 + \cdots + a_K = n)$$

We need to count all terms $y_1 y_2 \cdots y_n$ equal to $x_1^{a_1} x_2^{a_2} \cdots x_K^{a_K}$.
However each such term corresponds to a linear order elements of a multiset with K types, and there are a_i elements of type i $(i = 1, \ldots, K)$.
We already showed that this orderings counted by

$$\frac{n!}{a_1!\, a_2! \cdots a_K!} = \binom{n}{a_1, a_2, \ldots, a_K}. \qquad \square$$

Example 4.11. Prove that

$$\binom{n}{a_1, a_2, \ldots, a_K} = \binom{n}{a_1} \cdots \binom{n - a_1 - \cdots a_i}{a_{i+1}} \cdots \binom{n - a_1 - \cdots - a_{K-1}}{a_K}$$

for any $a_1 + \cdots + a_K = n$.

Proof: LHS count all linear orderings of the multiset $S = \{1^{a_1}, 2^{a_2}, \ldots, K^{a_K}\}$.

To obtain such a linear order, we can choose first a_1 positions of 1 from n (there are $\binom{n}{a_1}$ ways), then a_2 positions of 2 from $n - a_1$ (there are $\binom{n-a_1}{a_2}$ ways).

Assume that we already chose positions for $a_1, \dots, a_i$; we choose next a_{i+1} positions for $i+1$ from $n - a_1 - \dots - a_i$ remaining positions.

(there are $\binom{n-a_1-\dots-a_i}{a_{i+1}}$ ways), and so on.

$\square$

3. Non-integer exponent.

Defn (generalized binomial coefficient)

Let m be any real number, and k a nonnegative integer. Then $\binom{m}{0} = 1$ and

$$\binom{m}{k} = \frac{m(m-1)\dots(m-k+1)}{k!} \quad \text{if } k > 0.$$

Theorem. Let m be any real number. Then

$$(1+x)^m = \sum_{k \geq 0} \binom{m}{k} x^k.$$